The Logic of Backward Induction

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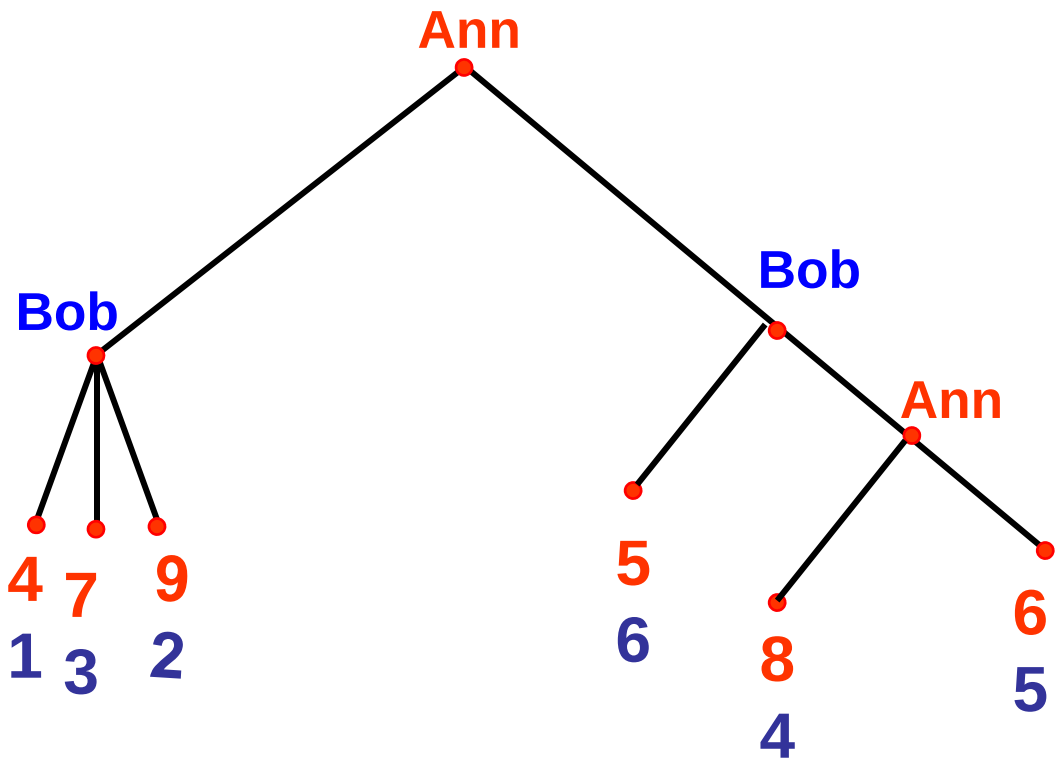
A Perfect Information Game



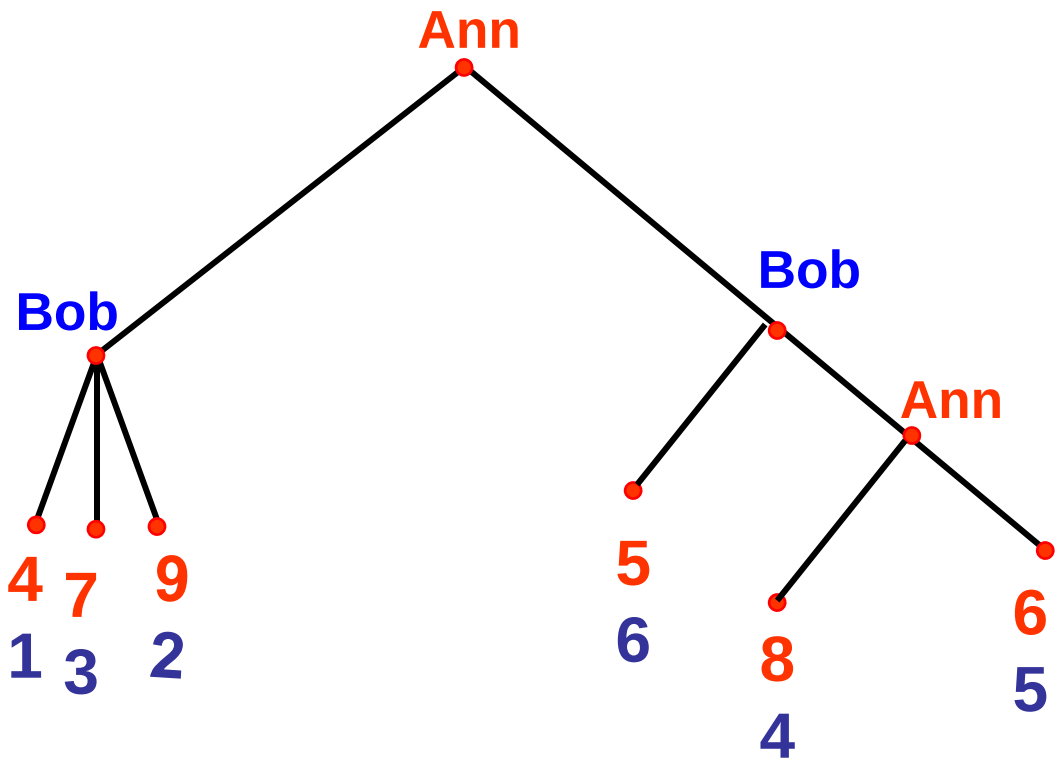
Reminder:

Backward Induction (BI)

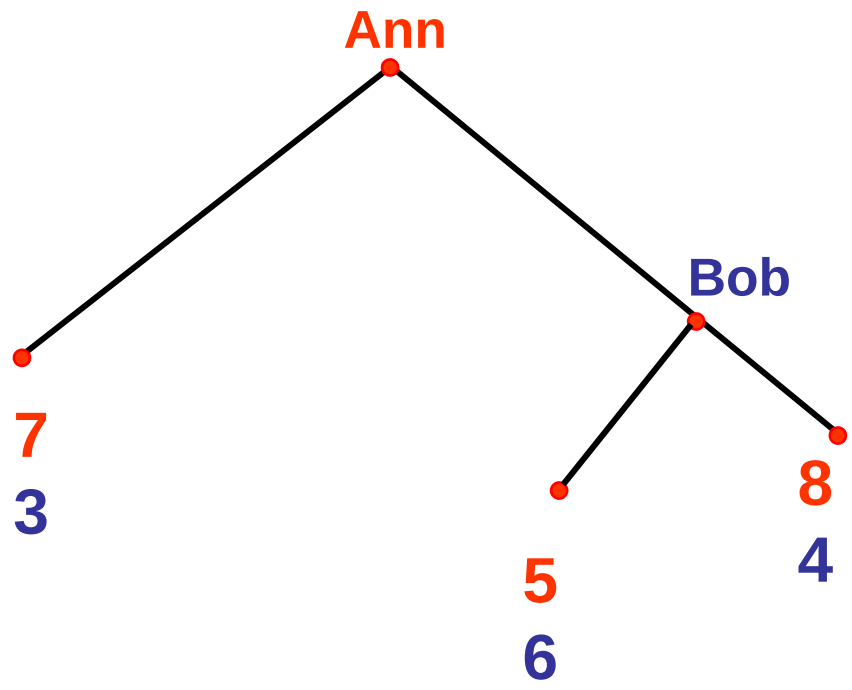
(For **generic** perfect information (PI) games):

- At a preterminal node *h*, the player at that node chooses the leaf that is best for him.
- The node *h* is replaced by the chosen leaf.
- The process is repeated until only one leaf remains.

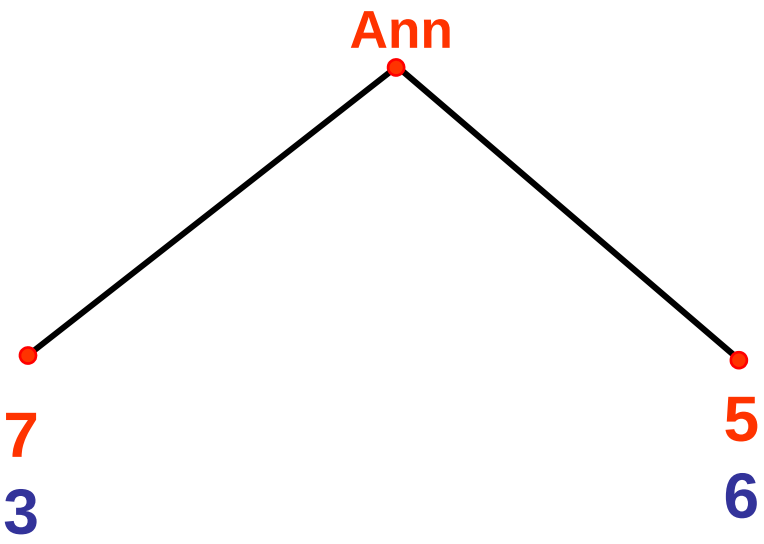
A Perfect Information Game



Backward Induction - 1



Backward Induction - 2



Backward Induction - 3

Ann chooses left, and the outcome is (**7**,**3**).

The Intuition for Backward Induction

- The last player, who must choose between leaves of the game tree, chooses an action that maximizes his payoff. Taking this as given, the previous player maximizes *his* payoff; and so on, until the beginning of the game is reached.
- This reasoning has been strongly criticized. We here examine its epistemic foundations.
- To make it work, simple rationality (payoff maximization) is not enough; the players must also ascribe rationality to each other. That is, *common knowledge* of rationality (ckr) should be assumed.
- But ckr appears to involve an inner contradiction: it depends on what players "would" do at nodes they *know* are unreachable under ckr.

So, let's substitute **strong belief** for knowledge.

- Defs. A player at a node *v strongly* believes a proposition *p* if he believes it (with prob. 1), unless it is logically incompatible with *v* being reached.
- *p* is *commonly* strongly believed if it is true, all players strongly believe it, all strongly believe that, all strongly believe *that,* and so on.

- Theorem. Common strong belief of rationality *(csbr)* is consistent, and implies that the backward induction outcome is reached.

Rationality

- Preliminary Observation: Backward Induction reasoning is intrinsically local. Players optimize only their actions at each node, not their strategies. Indeed, the concept of a "player" with several decision nodes plays no role in BI; the reasoning is unchanged if each player *i* is split into several independent "agents," one for each of *i*'s nodes, each with the same payoff as *i*.

- Definition. An agent is *rational* if his action is not *belief-dominated;* i.e., if he has no action that he believes yields him a higher payoff, no matter what is done subsequently. *Rationality* obtains if all agents are rational.

Battigalli and Sinischalchi

The notion of strong belief is due to Battigalli and Sinischalchi [2002]. They used it to formulate (and prove) a result that superficially resembles ours, but actually differs from it essentially. In fact, the BS result refers to the rationality of strategies rather than actions, provides a basis for forward rather than backward induction, and lies far deeper.

Recapitulating the theorem

Theorem. Common strong belief of rationality *(csbr)* is consistent, and implies that the backward induction outcome is reached.

Outline of Proof

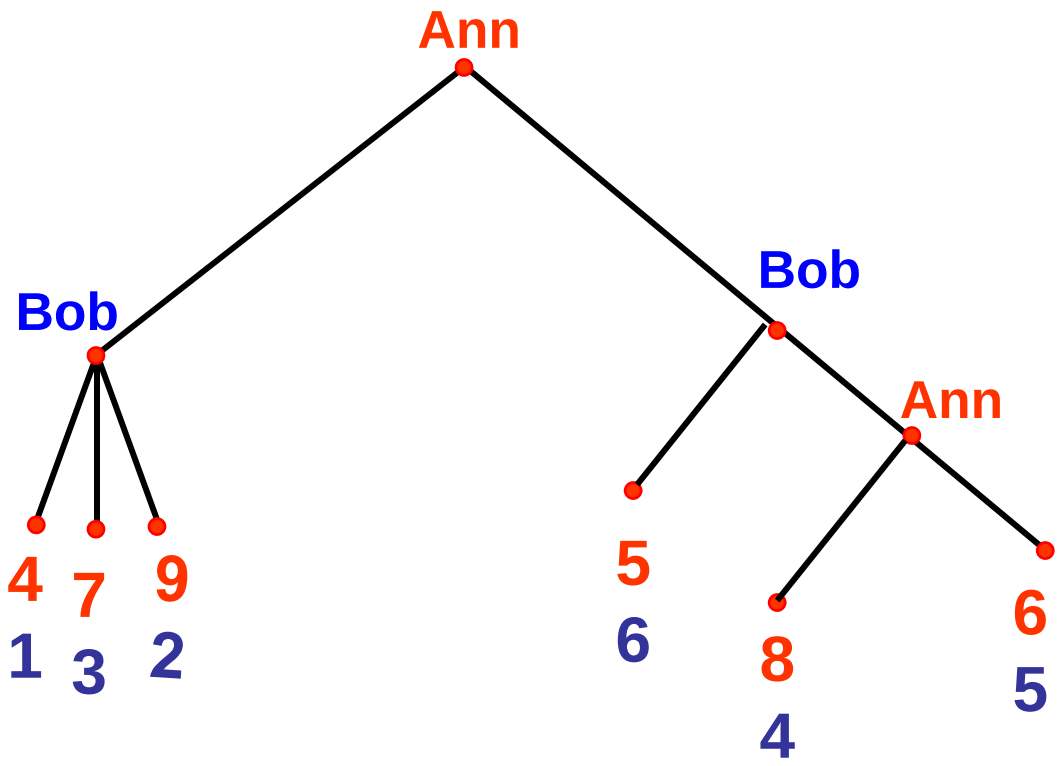
The proof has three parts. The first describes a process of successively eliminating actions (and the subsequent branches of the tree), called the Pruning Process (PP). The second identifies the outcome of the PP with that of common strong belief of rationality (*csbr*). Finally, the third part shows that the PP leads to the BI outcome. Symbolically:

$$csbr \leftrightarrow \text{PP} \leftrightarrow \text{BI}$$

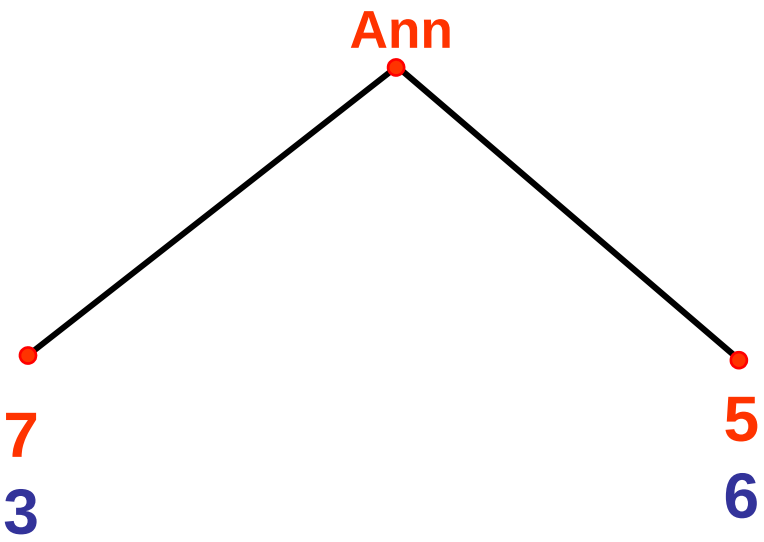
Part 1: The Pruning Process (PP)

Eliminate all dominated actions (and the subsequent branches), then iterate.

A Perfect Information Game



Pruning Process – 1



Pruning Process - 2

The right branch is pruned; the outcome is (**7**,**3**).

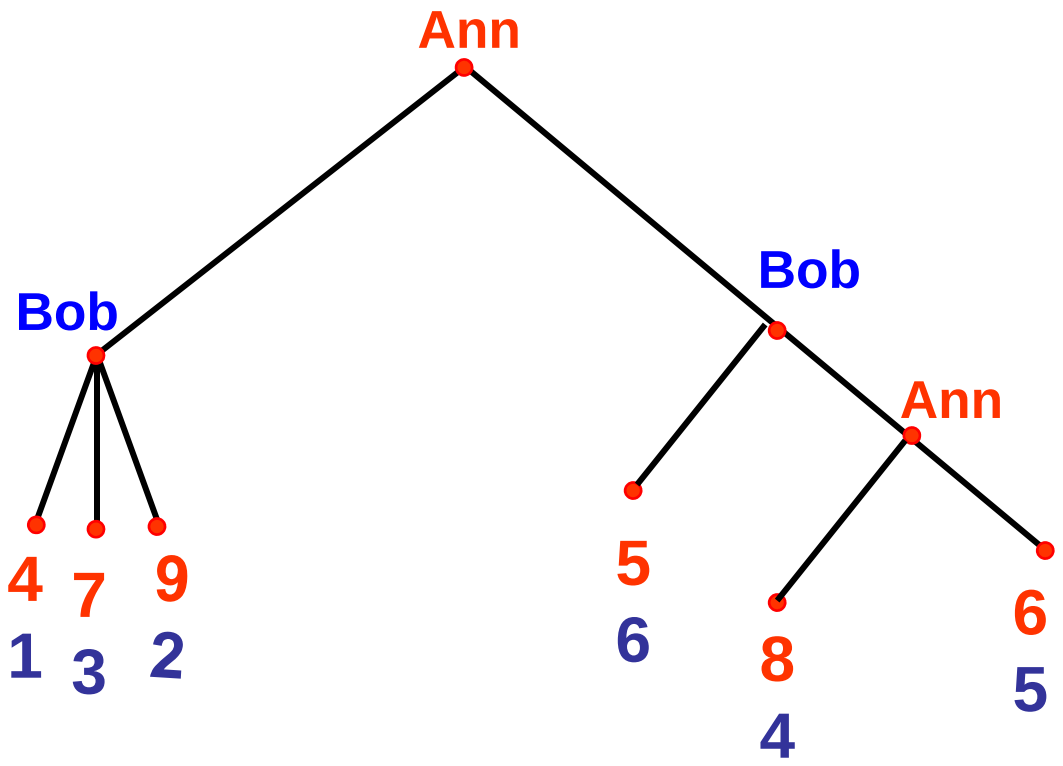
Part 2: *csbr* ↔ PP

Recall that *csbr* means that

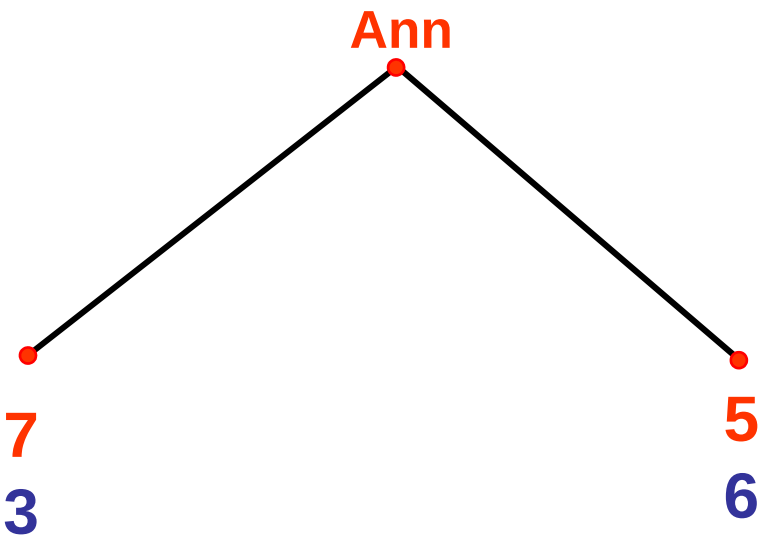
- all players are rational, and
- all players strongly believe the foregoing, and
- all players strongly believe the foregoing, and
- ...

  When one spells out exactly what these bullets say, they turn out to correspond exactly to the successive stages of the PP. ☺

A Perfect Information Game



rationality



Rationality and
Strong Belief of Rationality

Ann chooses left, and the outcome is (**7**,**3**).

Part 3: The PP leads to the BI outcome

- Label each node by the leaf that replaces it in the BI process.
- Then the label of a node *h* does not change when an action at *h*, and the subsequent branch, is pruned.
- So the root's label—which is the BI outcome—survives the PP.
- The PP must lead to a unique leaf.
- So that leaf is the BI outcome.

Semantic vs. Syntactic Formalisms

- The notion of "logical impossibility" is an essential element of strong belief, and so of our result.
- Semantic formalisms—states of the world, partitions, etc.—are commonly used in economics and game theory.
- Indeed that is what BS use.
- To express "logical impossibility" semantically, one needs the notion of a *complete* state space, which involves a non-trivial construction.
- We use a syntactic formalism: a formal language, axioms, deduction rules, theorems, etc.
- This enables us to express "logical impossibility" simply and naturally.

谢谢

Thank you!