Strategic Plays in Economic Equilibrium and its Applications

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Algorithm of Sharing

- Social Optimization
    - Decision Making to Maximize Social Welfare
- Market Equilibrium
    - Individual Optimization and Market Clearance
- Computational Economics
    - Submission of Private Information for the Delivery of Price and Allocation.

The Computational Model

- Computational Difficulties
- Dynamics and Eventual Convergence
- Data Quality: Truthfulness

A Divisible Goods Market

1. Truthful reporting is optimum for all agents only in specific cases.
2. Example: linear markets of two buyers and one seller with two goods.
    - utility functions of buyers: $u_1(x, y) = x + y$ and $u_2(x, y) = y$.
    - initial cash endowment of buyers: $e_1 = 1$, $e_2 = 1$.
    - seller has one unit of each goods.
3. Market equilibrium: seller set price to be $(1, 1)$. buyer 1 gets item 1 and buyer 2 get item 2.

Buyers' Cheating Strategies

1. Cheating is possible.
    - Buyer 1 reports: $u'_1(x, y) = \epsilon x + y$ where buyer 2 remains truthful $u_2(x, y) = y$.
    - initial cash endowment of buyers: $e_1 = 1$, $e_2 = 1$.
    - seller has one unit of each goods.
2. Market equilibrium: seller set price to be $(0, 2)$. buyer 1 gets $(1, 0.5)$ and buyer 2 gets $(0, 0.5)$

Utility Reporting Game Nash Equilibrium

1. Notations
   - Agent i has a utility $u_i(x, y) = a_i x + b_i y$, $i = 1, 2$
   - Agent i bids a utility $u'_i(x, y) = a'_i x + b'_i y$, $i = 1, 2$
2. Solution Concepts
   - Nash Equilibrium: None of Agent i, $i = 1, 2$, can increasing its true utility by switching from its report of (a'_i, b'_i).
   - Truthful Auction: It is a Nash Equilibrium for every agent to bid true utility.
3. Theorem(Adsul et al.): Truthful is a pure Nash equilibrium if and only if all utilities are the same in the linear market.

What Happens in Reality?

The Best Possibility: All three happen at the same time.

1. Social Justice
2. Market Equilibrium
3. Utility Bidding Nash Equilibrium

Main Content

Bandwidth Sharing is Technically Possible

Sharing with Money and No-Money

- Bandwidth provider gets paid for: For the sake of convenience?
- Sharing without money: How to ensure fairness?

Bandwidth Sharing in Practice

- OpenGarden
  - Internet everywhere ... Seamlessly share Internet. Connect all your devices without having to tweak any settings
- Principles in Sharing
  - Fairness: Is everyone fairly treated?
  - Truthfulness: Would everyone participate honestly?

From Fair to Truthful

- Fair Protocol
  - How to treat everyone fairly.
  - It has been an issue of past studies
- Truthful Protocol:
  - Would everyone willingly tell the truth.
  - The issue is under active studies in various problems in network economics.

Fair Protocol

- Principle of Peer2peer network: A peach for a plum.
- How many peaches for a plum?
- A Gbit for a Gbit?
- Half of mine for half of yours?

Truthful Protocol

- Can we protect the objectives of the designer from manipulations ?
- Truthful: It is to the benefit of every participant to tell the truth to the protocol.

A Peer-to-Peer Network Model and How to Share?

Define a P2P Network Model

Undirected Graph $G = (V, E; w)$

- V: nodes of the network. Each is owned by an agent.
- E: communication edges between two agents.
- $w : V \to N$: $w(u)$ the upload bandwidth of u to be allocated to its neighbours. W.l.o.g., $w(\cdot)$ is an integer.

Fairness: Proportional Response Bandwidth Sharing Protocol (Wu and Zhang 2007)

- Provide each a share of mine in proportion to what I receive from others.
- Let a_i, $i = 1, 2, \cdots, n$, be what I receive from others
- Let w be what I am going to give out
- I will give agent i, a total bandwidth $= w * \frac{a_i}{\sum_{j=1}^{n} a_j}$

Market Equilibrium

- Each has an item: Node u has a bandwidth weight w_u.
- The bandwidth of each node will have a (different) price: the price of w_u is p_u.
- Every node agent wants as much bandwidth as possible from others.
- Market clearance: Bandwidth of agent u is sold out or $p_u = 0$.

Mark Solution: Allocation and Pricing

- Allocation: The agent u allocation an x_{uv} portion of its bandwidth to agent v, if $(u, v) \in E$.
- The bandwidth of an agent is priced (individually): the price of w_u is p_u.

Individual Optimality

Given price vector p
Max $\sum_{v \in \Gamma(u)} x_{vu} w_v$
subject to $\sum_{v \in \Gamma(u)} x_{vu} p_v \leq p_u$, $\forall v : x_{vu} \geq 0$

- $x^u = \{x_{vu} : (v, u) \in E\}$ represents the percentages of bandwidth bought by node u from each of its neighbours.
- The utility is the total volume of bandwidth bought $\sum_{v \in \Gamma(u)} x_{vu} w_v$
- Budget constraint: $\sum_{v \in \Gamma(u)} x_{vu} p_v \leq p_u$.

Market Clearance

- All the bandwidth is sold out.
- $\forall u : \sum_{v \in \Gamma(u)} x_{uv} = 1$
- which is a global constraint.

An Example

- Three agents $u = 1, 2, 3$ with weight $w(u) = 10^u$.
- Network: each is connected to other, $\forall u \neq v : (u, v) \in E$
- How do we find an market equilibrium?

A General Solution

- Linear market: all utilities are represented as a linear function of allocations.
- Solution always exists.
- Can be re-written as a convex programming problem
- Polynomial time algorithm exists.

Network Bottleneck Decomposition

The Inclusive Expansion Threshold Ratio

- Threshold Ratio: $\alpha(G) = \min \frac{w(\Gamma(B))}{w(B)}$
- Maximal bottleneck, B with the minimum threshold ratio and the maximum size subset B.
- Example: $u = 1, 2, 3$, $w_u = 10^u$. B=$\{3\}$ is the maximal bottleneck.

Bottleneck Decomposition

- For $G = (V, E; w)$, set $G_1 = G$ and $i = 1$.
- while $(G_i \neq \emptyset)$ do
  - $\alpha_i = \alpha(G_i)$, B_i be the maximal bottleneck of G_i. $C_i = \Gamma_{G_i}(B_i)$.
  - $G_{i+1} = G_i - [B_i \cup C_i]$
  - $i++$ & $k = i + 1$.
- return $\mathcal{B} = \{(B_1, C_1), (B_2, C_2), \cdots, (B_k, C_k)\}$ for $i \geq 1$.
- Notation: $V_1 = V$, $V_{i+1} - V_i - (B_i \cup C_i)$.
- Example: $G = K_3$, $u = 1, 2, 3$, $w_u = 10^u$. $\mathcal{B} = \{(\{3\}, \{1, 2\})\}$

Bottleneck Decomposition: An Example

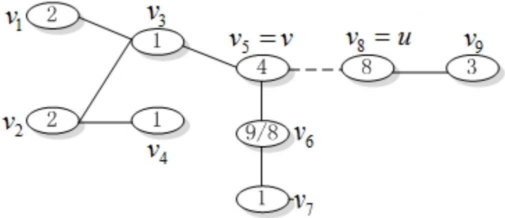


Figure : Number in each circle represents the weight of the vertex.

- $B_1 = \{v_1, v_2\}$, $C_1 = \{v_3, v_4\}$; $B_2 = \{v_8\}$, $C_2 = \{v_5, v_9\}$ and $B_3 = \{v_6\}$ and $C_3 = \{v_7\}$
- $\alpha_1 = 1/2$, $\alpha_2 = 7/8$ and $\alpha_3 = 8/9$.

Bottleneck Decomposition: Another Example

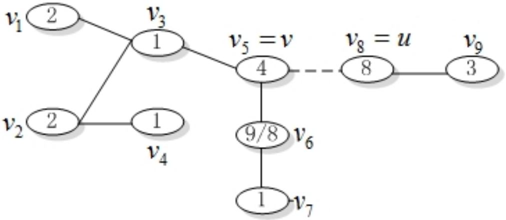


Figure : Number in each circle represents the weight of the vertex.

- If edge (v_5, v_8) is deleted, then in the new bottleneck decomposition of G', $B_1' = \{v_1, v_5, v_7\}$, $C_1' = \{v_3, v_6\}$. $B_2' = \{v_8\}$, $C_2' = \{v_9\}$. $B_3' = \{v_2\}$, $C_3' = \{v_4\}$
- $\alpha_1' = (1 + \frac{9}{8})/7 = 17/56$, $\alpha_2' = 3/8$ and $\alpha_3' = 1/2$.

Derived Market Equilibrium

- Find the bottleneck decomposition $\mathcal{B} = \{(B_1, C_1), (B_2, C_2), \cdots, (B_k, C_k)\}$ with $\alpha_1, \alpha_2, \cdots, \alpha_k$.
- B_i provides all bandwidth to C_i and vice versa.
- Solution can be found by the maximum flow algorithm
- Example: $G = K_3$, $u = 1, 2, 3$, $w_u = 10^u$. $\mathcal{B} = \{(\{3\}, \{1, 2\})\}$
  - Agent 1 gives all its 10 bandwidth to Agent 3
  - Agent 2 gives all its 100 bandwidth to Agent 3
  - Agent 3 gives its $\frac{1000}{11}$ bandwidth to Agent 1
  - Agent 3 gives its $\frac{10000}{11}$ bandwidth to Agent 2

Representing Utilities in Market Equilibrium

Incentive Ratio

1. Truthful reporting is possible if buyers do not seek improvement by less than a factor of r.
2. Matching bounds for the following markets:
    - 2 for Leontief market (with Chen and Zhang)
        - Utility $= \max\{\frac{x_i}{a_i}, i = 1, 2, \cdots, m\}$.
    - 2 for Linear Market and $e^{1/e} \approx 1.44$ for Cobb-Douglas market (with Chen, Zhang and Zhang)
        - Utility $= \{\Pi_{i=`}^{n} {x_i}^{a_i}\}^{1/\sum_{i=1}^{n} a_i}$.
    - 2 for WGS utilities.

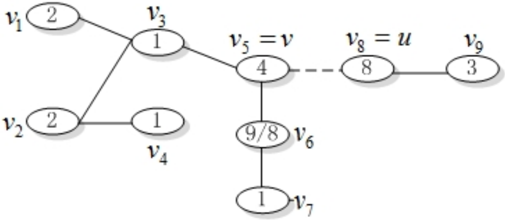
Incentive Issues in Bandwidth Sharing

- OpenGarden? Would everyone willing participate in the protocol?
- OpenGarden? Would someone take the advantage of the protocol?
- Our discussion tries to address the issue of incentives in network protocols.

Structural Cheating

- Can some agent cuts its connection to another agent?
- A realistic cheating act.
- Would one gains from doing it this way?
- It is impossible in the Example: $G = K_3$, $u = 1, 2, 3$, $w_u = 10^u$. $\mathcal{B} = \{(\{3\}, \{1, 2\})\}$

Example: Upload Bandwidth Change by Edge-cut



- In the above graph
  - $(\{v_1, v_2\}, \{v_3, v_4\})$; $(\{v_8\}, \{v_5, v_9\})$ and $(\{v_6\}, \{v_7\})$
  - $\alpha_1 = 1/2$, $\alpha_2 = 7/8$ and $\alpha_3 = 8/9$.
  - v_5 gets $w_5/\alpha_2 = 32/7$
- After deleting edge (v_5, v_8)
  - In the new bottleneck decomposition of G', $B'_1 = \{v_1, v_5, v_7\}$, $C'_1 = \{v_3, v_6\}$. $B'_2 = \{v_8\}$, $C'_2 = \{v_9\}$. $B'_3 = \{v_2\}$, $C'_3 = \{v_4\}$
  - $\alpha'_1 = (1 + \frac{9}{8})/7 = 17/56$, $\alpha'_2 = 3/8$ and $\alpha'_3 = 1/2$.
  - v_5 gets $4 * \alpha_1 = 4 * 17/56$.
- v_5 gets less, so does v_8. None of v_5 and v_8 improves.

Overview of Results

Relationships of Results

- Bottleneck Decomposition Derives a Market Equilibrium.
- Bottleneck Decomposition can be constructed in polynomial time.
- One cannot increase its threshold ratio in the bottleneck decomposition by removing an adjacent edge (or many).

Main Theorem

- No node agent can increase the amount of upload bandwidth by remove an edge or more under the proportional response mechanism.

Road Map for a Proof

- B_i, $i = 1, 2, \cdots, k$ are independent node sets.
- The bottleneck decomposition immediately restricts the feasibility of possible configurations of $\mathcal{B}' = ((B_1'.C_1'); (B_2', C_2'); \cdots)$.
- The incentive analysis of the agents u, v on broken link (u, v) further eliminates many other structures of the bottleneck decompositions.
- Dealing with extreme cases: neither of u and v can make an improvement of utility in the remaining structural possibilities.
- Main technique: Dense kernel removal.

Notations (i_u, j_u)

Let $(u, v) \in E$ be cut to obtain $G' = G - (u, v)$. If u and v are in different classes, w.l.o.g, we assume v is in C-class.

- Let vertex u appear in pair (B_l, C_l) at stage $l = i_u$ of the bottleneck decomposition of G.
- Similarly, let vertex u appear in pair (B'_l, C'_l) at step $l = j_u$ of the bottleneck decomposition of G'.
- Define $j_* = \min\{j_u, j_v\}$.

Dense kernel removal

- Goal: Derive a contradiction to the minimality of the α-ratio α_i of a pair (B_i, C_i) in the bottleneck decomposition
- Prove $\frac{w(C)}{w(B)} > \alpha_i$ for a pair (B, C) where $B \subseteq B_i$ and $C \subseteq C_i$.
- Remove the pair (B, C) from (B_i, C_i) to render a pair (B_i^c, C_i^c) with a smaller inclusive expansion ratio $\frac{w(C_i^c)}{w(B_i^c)} < \alpha_i$ and hence a contradiction.
- Denote it by $(B_i^c, C_i^c) = DKR(B, C; B_i, C_i)$.

Structural Feasibility Analysis

Basic Lemma: case where t is small

Basic Lemma

For the bottleneck decompositions $\mathcal{B}$ and $\mathcal{B}'$ of G and $G' = G - (u, v)$:

1. $(B'_t, C'_t) = (B_t, C_t)\ \forall t : 1 \leq t < j_*$.
2. $V'_t = V_t$, $\forall t : 1 \leq t \leq j_*$.
3. If $V'_{j_*} = V_{j_*}$, then $\alpha'_{j_*} \leq \alpha_{j_*}$.
4. $\forall t < j_* : B'_t \cap (\cup_{i=1}^{k} C_i) = \emptyset$;
5. $\forall t < j_* : B_t \cap (\cup_{i=1}^{k} C'_i) = \emptyset$;
6. $j_* \leq i_v \leq i_u$.

Key Lemma: (B'_t, C_i) cases

Key Lemma

Consider the bottleneck decompositions $\mathcal{B}$ and $\mathcal{B}'$ of G and G'. Each of the following conditions implies that for any $1 \leq t \leq k'$ with $\alpha'_t < 1$, $B'_t \cap (\cup_{i=1}^{k} C_i) = \emptyset$:
1. for the case that $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, u and v are both in C'-class (Case 1);
2. for the case that $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$, $i = 1, \cdots, k$, v is in C'-class (Case 2), and
3. for the case that $(u, v) \notin B_i \times C_i$ (Case 3).

Main Lemma: (B_t, C_i') cases

Main Lemma

Consider the bottleneck decompositions $\mathcal{B}$ and $\mathcal{B}'$ of G and G'. We have that if the case that $(u, v) \in B_i \times C_i$ and u, v are both in B'-class does not happen, then $B_t \cap (\cup_{i=1}^{k'} C_i') = \emptyset$ for any step t with $\alpha_t < 1$.

Incentive Analysis

Possible Structure

For one of the u and v to have the incentive to cheat (by deleting the edge (u, v)), we only have the following three cases need to be consider:

1. $u \in B_{i_u}$, $v \in C_{i_v}$, $v \in B'_{j_v}$ and $u \in B'_{j_u}$: imply that
    - $j_v < j_u$, $\alpha'_{j_v} = 1$ and $i_v = i_u$.
    - Agent u cannot cheat unless $\alpha'_{j_u} > \alpha_{i_u}$.
2. $u \in B_{i_u}$, $v \in C_{i_v}$, $v \in C'_{j_v}$ and $u \in B'_{j_u}$:imply that
    - $i_v \leq i_u$, $j_u < j_v$.
    - Agent u cannot cheat unless $\alpha_{i_u} < \alpha'_{j_u}$.
    - Agent v cannot cheat unless $\alpha_{i_v} > \alpha'_{j_v}$.
3. $u \in B_{i_u}$, $v \in B_{i_v}$, $v \in C'_{j_v}$ and $u \in B'_{j_u}$: imply that
    - $i_v = i_u = k$ and $\alpha_k = 1$.
    - Agent v will cheat if this case is truely possible.

The Rest of Extreme Cases

Case 1: $(u, v) \notin B_i \times C_i$, $i = 1, 2, \cdots, k$

The main idea is to show that the bottleneck decomposition of G' will remain the same as that of G unless the utility of u or v is reduced. This is proved by the following steps:

Lemma

If a vertex w is a C-class vertex in C_j, it cannot belong to B'_1.

Then it follows that

Lemma

If $(u, v) \notin B_i \times C_i$, $i = 1, 2, \cdots, k$, then $(B'_1, C'_1) = (B_1, C_1)$. And further $(B'_l, C'_l) = (B_l, C_l)$, $l = 1, 2, \cdots, k$.

By the proposition, each vertex's utility is exactly determined by its α-value and its class. Since the bottleneck decomposition of G' is same to that of G, then we can conclude the result.

Case 2: $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$

For this extrem case, firstly we introduce a property.

property

Given an edge $(u, v) \in B_i \times C_i$, $i = 1, 2, \cdots, k$. If $G' = G - (u, v)$, then in the bottleneck decomposition of G', $\min\{j_u, j_v\} \leq i = \min\{i_u, i_v\}$.

Now we can divide the case into three sub cases, i.e. $j_v < j_u$ $j_v = j_u$ and $j_v > j_u$.

Notations and SubCase 2: $j = j_v < j_u$

- Properties:
    - $\alpha'_{j_v} < 1$ and $j \leq i$
    - $\forall t < j : (B'_t, C'_t) = (B_t, C_t)$
    - $V_t = V'_t$ for $t \leq j$, where $V_t = V - \cup_{k=1}^{t-1}(B_k \cup C_k)$
- Feasibility Requirement: v must be B'-class.
    - No C-class vertex is in B'_{j_v} by Key Lemma & $v \in C'_{j_v}$.
    - Next, we discuss two cases: $j_v < i$ and $j_v = i$.
- Incentive analysis of u and v: None increases its utility.

Feasible Configurations (if $j < i$): Prove $v \in B'_j$

1. Partition B'_j as $B'_{j1} = \bigcup_{l=j+1}^{k} \left(B'_j \cap B_l \right)$, $B'_{j2} = B'_j \cap B_j$.
2. As $V'_j = V_j$, $\Gamma(B_l) \cap V_l \subseteq V_j$, $\forall l \geq j$. Thus, $\Gamma(B'_j \cap B_l) \cap C_l \subseteq C'_j$.
3. $C'_{j1} = \bigcup_{l=j+1}^{k} \left[\Gamma(B'_j \cap B_l) \cap C_l \right]$ and $C'_{j2} = C'_j \setminus C'_{j1}$.
4. As $v \in C'_j \cap C_i$, $\exists x \in B'_j$, $(x, v) \in E$. Then, by Key Lemma, $x \in B_l$, for some $l \geq i > j_v$.
5. $B'_{j1} \neq \emptyset$ and $C'_{j1} \neq \emptyset$.
    - $\frac{w(C'_{j1})}{w(B'_{j1})} \geq \alpha_{j+1} > \alpha_j \geq \alpha'_j$
    - $\frac{w(C'_{j_v 2})}{w(B'_{j_v 2})} < \alpha'_{j_v}$
6. No neighbor in C'_{j1} for any vertex in B'_{j2}.

So $\Gamma(B'_{j2}) \cap V'_j \subseteq C'_{j2}$ and a new pair $(B'_{j2}, \Gamma(B'_{j2}) \cap V'_j)$ whose α-value is strictly less than α'_j. This is a contradiction.

Feasible Configurations (if $j = i$): Prove $v \in B'_j$

To the contrary, let $v \in C'_j$.

1. A similar proof combining the above case and the case $j_v = j_u = i$.
2. More details for the other cases will be presented in the paper...

Related Works

Convergence

- Wu and Zhang (STOC 2007): The fairness solution converges to the economic solution of the market equilibrium.
- Interpretation: Fairness and Commercial Solution Matches.

Truthful Property of Our Result

- No agent can report a broken link to benefit in the market solution.
- Interpretation: Market equilibrium converges with utility bidding game Nash equilibrium.

Incentive Ratio

- The linear market equilibrium is not truthful. (Adsul, et al., SAGT 2010).
- Each agent may cheat to increase its utility, maximum twice as much and tight (Chen, et al., ICALP 2012).

Why our agent is not able to lie under the linear market?

- Cutting an edge can be realistic.
- The act changes one's linear utility, and also that of its neighbour.

Remarks

- Is it possible to extend to general cases?
- Agent can cheat their utility functions.
- The incentives may not be the same as the simple general utility function settings in Economics
- Network protocol design opens up new issues in algorithmic game theory.